Chapter 7. Relativistic Effects

7.1 Introduction

This chapter details an investigation into relativistic effects that could cause clock errors and therefore position errors in clock coasting mode. The chapter begins with a description of relativistic corrections that are already taken into account for the GPS satellites. Though these corrections are known and implemented [ICD-GPS-200, 1991], no similar corrections are made for GPS receivers that are in motion. Deines derived a set of missing relativity terms for GPS receivers [Deines, 1992], and these are considered in great detail here. A MATLAB simulation was used to predict the relativity effects based on the derivation of Deines, and flight data were used to determine if these effects were present.

7.2 Relativistic Corrections for the GPS Satellite

The first effect considered stems from a Lorentz transformation for inertial reference frames. This effect from special relativity is derived from the postulate that the speed of light is constant in all inertial reference frames [Lorentz et. al., 1923]. The correction accounts for time dilation, i.e. moving clocks beat slower than clocks at rest. A standard example is as follows [Ashby and Spilker, 1996]. Consider a train moving at velocity \( v \) along the \( x \) axis. A light pulse is emitted from one side of the train and reflects against a mirror hung on the opposite wall. The light pulse is then received and the round trip time recorded. If the train car has width \( w \) (see Fig. 7.1), then the round trip time according to an observer on the train is:
Figure 7.1 Illustration of Time Dilation Using Light Pulses
\[ t_{\text{train}} = \frac{2w}{c} \quad (7.1) \]

where \( c \) is the speed of light. Now consider an observer that is stationary with respect to the train tracks. The train has moved a distance \( vt_{\text{track}} \) between emit and receive times according to this observer's clock. Thus, the total time elapsed in the stationary frame is given by the following relation:

\[ t_{\text{track}} = 2 \sqrt{w^2 + \left(\frac{1}{2}vt_{\text{track}}\right)^2} \quad \left(\frac{c}{c}\right) \quad (7.2) \]

This yields:

\[ t_{\text{track}} = \frac{2w}{c \sqrt{1 - \frac{v^2}{c^2}}} \quad (7.3) \]

Implicit in this derivation is the assumption that the speed of light is the same in both reference frames. This is one of Einstein's postulates and was verified by the Michelson-Morley experiment [Krane, 1983]. Thus, the moving clock beats slower than the stationary clock as measured by a stationary observer:

\[ \frac{t_{\text{track}}}{t_{\text{train}}} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \quad (7.4) \]

Using a binomial expansion and omitting the higher order terms:
This represents one clock correction term that must be applied for GPS satellites. The clocks in orbit experience time dilation and must therefore be adjusted in order to maintain agreement with clocks on the surface of the Earth. The orbital velocity of the GPS satellite is determined from Kepler's Third Law [Pratt and Bostian, 1986]:

\[ T^2 = \frac{4\pi^2}{GM_e} a^3 \]  \hspace{1cm} (7.6)

where:
- \( a = 26561.75 \) km, the semimajor axis of the GPS orbit
- \( GM_e = 3.986005 \times 10^{14} \) m\(^3\)/s\(^2\), the Earth gravitational constant
- \( T \) is the orbital period (seconds)

From this we find that \( T = 43,082 \) s. Assuming circular orbits the satellite velocity is:

\[ \nu = \frac{2\pi a}{T} \]  \hspace{1cm} (7.7)

\[ = 3873.8 \text{ (m/s)} \]

Now the clock drift due to time dilation is calculated as:

\[ -\frac{\nu^2}{2c^2} = -0.83 \cdot 10^{-10} \text{ (s/s)} \]  \hspace{1cm} (7.8)

In one day this drift would cause a clock offset of -7.2 µs.
Another correction must be made to account for the height of the GPS satellites above the surface of the Earth. With an orbital altitude of about 20,200 km the satellite clock experiences a gravitational potential that is significantly different from that experienced by a clock located on the surface of the Earth. This causes a gravitational frequency shift in the GPS carrier as the signal travels from the satellite to an antenna on or near the surface of the Earth [Spilker, 1978]. The result is that the satellite clock appears to run faster to an observer on the Earth than it would to an observer at the satellite.

If the Earth is considered to be spherically symmetric, the gravitational potential $\phi(r)$ can be approximated as [Nelson, 1990]:

$$\phi(r) = -\frac{GM_e}{r}$$  \hfill (7.9)

where $r$ is the radial distance from the center of the Earth. The gravitational effect is thus [Hofmann-Wellenhof, 1994]:

$$\frac{\Delta \phi}{c^2} = \frac{\phi(R_e+h) - \phi(R_e)}{c^2} = -1.67 \cdot 10^{-10} - (-6.95 \cdot 10^{-10})$$  \hfill (7.10)

$$= 5.28 \cdot 10^{-10} \text{ (s/s)}$$

where: $R_e = 6,378$ km, the equatorial radius of the Earth

$h = 20,184$ km, the altitude of the GPS satellite

This clock drift accumulates to 45.6 µs after one day.
The combination of gravitational frequency shift and time dilation results in a satellite clock drift for observers on the Earth [Deines, 1992]:

\[ \tau = \int \left( 1 + \frac{\Delta \phi}{c^2} - \frac{v^2}{2c^2} \right) dt \]  

(7.11)

where:
- \( \tau \) is proper time of the clock carried by the GPS satellite
- \( t \) is coordinate time in the inertial reference frame
- \( \Delta \phi \) is the gravitational term
- \( v \) is the satellite velocity
- \( c \) is the speed of light

The combined effect is a drift of \( 4.45 \times 10^{-10} \) s/s which would cause a clock offset of 38.4 µs after one day. Therefore, the satellite clocks are tuned so that the observed frequency on Earth is 10.23 MHz:

\[ \frac{f_{\text{obs}} - f_{\text{tr}}}{f_{\text{obs}}} = 4.45 \cdot 10^{-10} \]  

(7.12)

where:
- \( f_{\text{obs}} = 10.23 \text{ MHz} \), the clock frequency as observed on Earth
- \( f_{\text{tr}} \) = the clock frequency as observed at the satellite

Thus, the clocks are tuned to 10.22999999545 MHz before launch [Spilker, 1978]. This may seem like a minor correction, but a 1 µs clock error is roughly equal to 300 m in terms of distance. A 1 µs error would build up in 38 min if the relativistic effects of Eq. 7.11 were not corrected.
The assumption of circular orbits is somewhat erroneous. GPS orbits typically have an eccentricity of less than 0.01 [Ashby and Spilker, 1996], but even the slightest eccentricity causes the satellite to change altitude periodically during orbit. This results in the need for an additional relativity term, separate from the terms of Eq. 7.11 [Nelson, 1991]:

\[
\Delta t_{\text{per}} = -\frac{2\sqrt{GM_e}a}{c^2}e\sin(E)
\]  

(7.13)

where:  
\(\Delta t_{\text{per}}\) is the periodic relativistic error in seconds  
e is the orbit eccentricity  
E is the eccentric anomaly  

For an eccentricity of 0.01, the maximum effect is ±22.9 ns, or ±6.86 m. The correction is added to the clock polynomial based on the broadcast coefficients for that satellite [ICD-GPS-200, 1991]:

\[
\Delta t_{\text{SV}} = a_{f_0} + a_{f_1}(t - t_{oc}) + a_{f_2}(t - t_{oc})^2 + \Delta t_{\text{per}}
\]  

(7.14)

where:  
\(\Delta t_{\text{SV}}\) is the SV PRN code phase time offset in seconds  
t is GPS system time  
t_{oc} is the clock data reference time  
\(a_{f_0}, a_{f_1}, a_{f_2}\) are polynomial coefficients broadcast by the satellite  

Thus, the complete satellite clock correction is as follows. The terms of Eq. 7.11 are corrected by setting the clocks to 10.22999999545 MHz. The polynomial coefficients \((a_{f_0}, a_{f_1}, a_{f_2})\) are used to correct for satellite clock bias, drift, and acceleration with respect to GPS time. The additional term, \(\Delta t_{\text{per}}\), accounts for GPS orbit eccentricity in the application of
Eq. 7.11. Prescriptions from special and general relativity are applied to correct known relativistic errors for the satellites.

Some scientists have observed small GPS pseudorange errors known as the "bowing effect" which are independent of what type of receiver is used [Klepczynski, 1986], and have theorized that these errors may be uncorrected relativistic effects [Deines, 1992]. Nelson (1991) has concluded that the relativity corrections as applied for GPS time transfer are not in need of modification, but that ephemeris errors could be the source of systematic errors. However, the large relativistic effects are clearly accounted for in the satellite clock corrections. The remainder of this chapter deals with relativistic corrections for GPS receivers on dynamic platforms. These effects are unaccounted for in GPS receiver algorithms.
7.3 GPS Receivers on Moving Platforms

The goal of this work is to analyze the effects that Deines claims are not accounted for in GPS algorithms. This claim has not been verified experimentally, which is the purpose of this study. In order to determine a flight profile that would maximize the predicted effects, an analysis was done which shows that two of the terms Deines claims are missing would be observable in the double difference (Eq. 3.16). Methods of verifying the presence of the relativistic effects based on errors in the double differences will be presented in Sec. 7.4.

In this section we seek to develop a method of testing for the presence of relativistic effects predicted by Deines. We begin by introducing the MATLAB simulation used to compute the predicted effects, and continue by using this simulation to check for agreement with examples given by Deines [Deines, 1992]. The flight profile and satellite positions were used in a MATLAB simulation to evaluate the relativistic effects. A prediction of relativistic effects for a GPS receiver undergoing acceleration in a turn is given, followed by a calculation of relativistic effects in differential GPS. Here, the relativistic effects are calculated both for the aircraft and the ground station, given a straight and level flight profile. The resulting errors that would be evident in the double differences are presented, from which a flight test experiment is devised.
7.3.1  Predicted Relativity Terms for GPS Receivers

The missing relativity terms that are predicted by Deines come from the following equation [Deines, 1992]:

\[
\tau \approx \int_T^t \left( 1 + \frac{\Delta \phi}{c^2} - \frac{v^2}{2c^2} \right) dt + \int_T^t \left( \frac{A \cdot \rho}{c^2} + \frac{V \cdot v}{c^2} - \frac{V^2}{2c^2} \right) dt \quad (7.15)
\]

where:
- \( \tau \) is proper time
- \( t \) is coordinate time
- \( T \) is the integration time
- \( \Delta \phi \) is a gravitational term
- \( V \) is the receiver velocity vector
- \( V \) is the receiver speed
- \( v \) is the satellite velocity vector
- \( v \) is the satellite speed
- \( A \) is the receiver acceleration
- \( \rho \) is a vector from the receiver to the satellite
- \( c \) is the speed of light

The first integral includes the satellite clock corrections that are currently implemented (Eq. 7.11). The terms in the second integral are the relativity terms that Deines claims are unaccounted for in GPS algorithms. During the remainder of this chapter we will be concerned primarily with the terms under the second integral of Eq. 7.15. These are represented here to avoid later confusion:

\[
\text{Missing Terms} = \int_T^t \left( \frac{A \cdot \rho}{c^2} + \frac{V \cdot v}{c^2} - \frac{V^2}{2c^2} \right) dt \quad (7.16)
\]
The third term in Eq. 7.16 is a time dilation term which is analogous to the time dilation term applied for the GPS satellite as shown in the first integral of Eq. 7.15. The time dilation term for the receiver does not depend on any satellite parameters. The other two terms presented by Deines vary depending on which satellite is considered, because \( v \) and \( \mathbf{p} \) are different for each satellite. To numerically calculate these terms, a GPS simulation can be used to provide satellite positions and a flight profile can be generated in MATLAB. First, consider the test cases presented by Deines which were used to illustrate that the predicted relativistic effects are significant.

In the examples given by Deines, a receiver is on board an aircraft at the equator. A GPS satellite passes directly overhead at an altitude of 20,200 km in an inclined orbit of 55°. The aircraft moves at a speed of 900 m/s in each of the four principal directions. Satellite positions from a FORTRAN based GPS simulation were used along with a simulated flight profile to numerically calculate the relativistic effects in MATLAB. Each case was then compared to the results given by Deines to check for agreement.

In the first example the aircraft flies east at a constant altitude and therefore has an acceleration as it follows the curvature of the Earth. The flight profile was generated in MATLAB with a time increment of one second. A GPS simulation provided satellite positions at one second intervals, and a satellite that is just passing overhead was chosen for the computation. The simulated flight began at a point directly under the satellite and continued eastward along the equator. For this case Deines predicted a clock drift of \( 8.9 \times 10^{-11} \) s/s which is
a combination of \(+6.56 \times 10^{-11} \text{ s/s for the } \frac{A \cdot \rho}{c^2} \) term and \(+2.34 \times 10^{-11} \text{ s/s for the } \frac{2V \cdot v - V^2}{2c^2} \) term.

The results of the MATLAB simulation for each term are shown in Figs. 7.2-7.3. The simulation agreed with Deines in magnitude but differed in sign for the \(\frac{A \cdot \rho}{c^2} \) term. This is due to the fact that \(A\) points toward the center of the Earth while \(\rho\) is a vector from the receiver to the satellite. Thus, the vectors are parallel but the directions are opposite which yields the negative result. The drift was not constant in the MATLAB simulations because the geometry between the receiver and the satellite changed slightly during the 100 second run time. The \(\frac{2V \cdot v - V^2}{2c^2} \) term is in agreement with Deines. The results of the MATLAB simulations were compared to Deines' results for each of the four cases. Table 7.1 shows the clock drifts presented by Deines, and Table 7.2 shows the results of the MATLAB simulations. Though the numbers are different the discrepancies between the two are minor. The simulation is accurate in the sense that it agrees with Deines' examples.

As noted, the \(\frac{A \cdot \rho}{c^2} \) term showed a difference in sign due to the interpretation of which way \(A\) and \(\rho\) point. Deines states that \(\rho\) is a vector from the receiver to the satellite. Clearly the centripetal acceleration of the aircraft points toward the center of the Earth. Thus, the \(\frac{A \cdot \rho}{c^2} \) terms from the MATLAB simulation are negative whereas Deines showed these terms to be positive.
Figure 7.2  First Term of Clock Drift from MATLAB Simulation for an Aircraft Flying East Along the Equator at 900 m/s with a GPS Satellite Directly Overhead
Figure 7.3  Second Term of Clock Drift from MATLAB Simulation for an Aircraft Flying East Along the Equator at 900 m/s with a GPS Satellite Directly Overhead
Table 7.1 Relativistic Effects Predicted by Deines for Test Cases at the Equator

<table>
<thead>
<tr>
<th>From Deines</th>
<th>(V_{\text{inertial}}) (m/s)</th>
<th>(\frac{A \cdot \rho}{c^2}) (s/s)</th>
<th>(\frac{2V \cdot v - v^2}{2c^2}) (s/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>EAST</td>
<td>1365</td>
<td>+6.56 x 10^{-11}</td>
<td>+2.34 x 10^{-11}</td>
</tr>
<tr>
<td>NORTH</td>
<td>1013</td>
<td>+3.61 x 10^{-11}</td>
<td>+3.01 x 10^{-11}</td>
</tr>
<tr>
<td>SOUTH</td>
<td>1013</td>
<td>+3.61 x 10^{-11}</td>
<td>-4.15 x 10^{-11}</td>
</tr>
<tr>
<td>WEST</td>
<td>435</td>
<td>+0.67 x 10^{-11}</td>
<td>-1.18 x 10^{-11}</td>
</tr>
</tbody>
</table>

Table 7.2 Relativistic Effects from MATLAB for Test Cases at the Equator

<table>
<thead>
<tr>
<th>From MATLAB</th>
<th>(V_{\text{inertial}}) (m/s)</th>
<th>(\frac{A \cdot \rho}{c^2}) (s/s)</th>
<th>(\frac{2V \cdot v - v^2}{2c^2}) (s/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>EAST</td>
<td>1365</td>
<td>-6.56 x 10^{-11}</td>
<td>+2.34 x 10^{-11}</td>
</tr>
<tr>
<td>NORTH</td>
<td>1013</td>
<td>-3.63 x 10^{-11}</td>
<td>+3.76 x 10^{-11}</td>
</tr>
<tr>
<td>SOUTH</td>
<td>1013</td>
<td>-3.63 x 10^{-11}</td>
<td>-2.60 x 10^{-11}</td>
</tr>
<tr>
<td>WEST</td>
<td>435</td>
<td>-0.67 x 10^{-11}</td>
<td>-1.18 x 10^{-11}</td>
</tr>
</tbody>
</table>
In the North and South Cases for the second term, Deines used \( 1013 \hat{N} (\hat{S}) \) rather than \( 900 \hat{N} (\hat{S}) + 465 \hat{E} \). This caused the \( \frac{2 \mathbf{V} \cdot \mathbf{v} - \nu^2}{2c^2} \) terms to be in disagreement for those two cases. However, the simulation is largely consistent with the work presented by Deines and can be used to predict relativistic effects for more general cases.

### 7.3.2 Example of an Aircraft Accelerating During a Turn

As an example, consider what happens when an aircraft is banking. The receiver acceleration vector becomes large, for instance 19.6 m/s\(^2\) for a 2g turn. This represents a 2-3 order of magnitude increase over the centripetal acceleration of an airplane traveling straight and level to the east at 100 m/s along the equator. In that case \( |\mathbf{A}| \) is only 0.05 m/s\(^2\). The relativistic effects predicted by Eq. 7.16 for a 2g turn are shown in Fig. 7.4.

In Fig. 7.3, the relativity terms for nine satellites in view are shown. Recall from the discussion of Eq. 7.16 that two of the terms proposed by Deines are different for each satellite. This results in an accumulated effect that depends on which satellite is used. During a 2g turn lasting 16 s, eight of the nine predicted relativistic effects reach 10 ns or more. For two satellites the error is as large as 40 ns, and for one satellite the error is nearly 60 ns. A 40 ns timing error represents a 12 meter range error to the satellite. However, errors of this magnitude are not experienced in practice. This indicates that some discretion should be shown in interpreting the acceleration vector. For the remainder of this chapter \( \mathbf{A} \) will be taken as the centripetal
Figure 7.4  Predicted Relativistic Effects for an Aircraft in a Simulated 2g Turn — Each Satellite in View Produces a Different Effect
acceleration pointing towards the rotation axis of the Earth. Based on this analysis, the flight profile to be considered for a flight test is straight and level. The next step is to determine the relativistic effects in DGPS for such a case, and determine if the effects proposed by Deines are large enough to measure.

7.3.3 Relativistic Effects in Differential GPS

A ground station is now included to show the impact of the relativistic effects on Differential GPS. In this case, both the ground station and airborne receivers are moving in the Earth Centered Inertial frame and therefore experience relativistic effects according to Deines.

As an example consider a ground station at 0° Latitude and 0° Longitude. The ground station velocity is 465 m/s in the ECI frame. The aircraft starts at the ground station and flies east at 100 m/s, resulting in a velocity of 565 m/s in the ECI frame. The difference between air and ground clock offsets is manifest in the carrier phase single difference (Eq. 3.14). This difference between the clock offsets is shown in Fig. 7.5 for a 100 second run time for each satellite in view. Again the satellite positions are taken from a FORTRAN simulation, and there are nine satellites in view. The result of differencing the relativistic effects between the ground and the air is a clock drift between the receivers that shows up in the single difference for a particular satellite. Again we note that the effect depends on which satellite is used because \( \mathbf{v} \) and \( \mathbf{p} \) are different for each satellite.
Figure 7.5  Predicted Relativistic Effects for Straight and Level Flight as Observed in the Single Differences — Each Satellite Produces a Different Effect
By choosing the highest elevation satellite as a reference, the double difference (Eq. 3.16) can be formed. Because the effects predicted by Deines are satellite dependent, a residual relativity effect is predicted for the double differences. In other words, the relativistic effects are not common for all channels, unlike temperature effects which were shown in Chapter 6 to affect each single difference by the same amount. This means that temperature effects would be removed from the double difference, but not uncommon errors like the relativistic effects proposed by Deines. Fig. 7.6 shows that significant errors are predicted to build up in the double difference over time. Here we have nine satellites in view, so there are eight double differences. Two of the double differences show errors that grow to about -0.34 ns, or -0.1 m over a short flight of only 100 s.

The results shown in Fig. 7.6 indicate that significant errors are predicted by Deines. Consider that a -0.1 m error is predicted in two of the double differences after only 100 s, which means that a longer flight would produce even larger effects. Analyzing Eq. 7.16 further, we find that the first two terms are largely dependent on the velocity of the aircraft and the time in flight. The magnitude of the centripetal acceleration is $|A| = \frac{V_{\text{inertial}}^2}{R}$, and the $\frac{V \cdot \gamma}{c^2}$ term is clearly dependent on receiver velocity. The implication is that a flight of 200 s at 50 m/s would produce very similar results to the accumulated relativity errors shown in Fig. 7.6. This means that a small aircraft traveling at moderate speeds (50-60 m/s) can be used to produce measurable effects, if they exist. Fig. 7.6 illustrates a 10 km flight which produces double difference errors as large as 0.1 m. Therefore, a 40 km flight should produce errors roughly four times as large.
Figure 7.6  Predicted Relativistic Effects for Straight and Level Flight as Observed in the Double Differences — the Highest Elevation Satellite is the Reference
Errors of that magnitude (0.4 m) in the double differences would be measurable. Thus, the results shown in Fig. 7.6 are the basis of the flight test experiment that was devised to test the theory of Deines.
7.4 Flight Test Experiment

7.4.1 Flight Test — Introduction

The purpose of this flight test was to measure relativistic effects as predicted by Deines in Eq. 7.16. The test is based on the results shown in Fig. 7.6, and the flight profile is straight and level flight over a distance of about 40 km. The relativistic effects predicted by Deines would build up in the double differences during the flight. Thus, we seek to measure an accumulated effect at the conclusion of the flight. Two methods will be used to verify the presence of the relativistic effects.

First, the double difference errors can be converted into position errors, and this can be compared with experimental position error. This requires knowledge of true position at the beginning and end of the flight test. The true position at the beginning of the flight can be used to initialize an L1 carrier phase ambiguity resolved solution. This solution is carried through the flight, and is compared to the true final position to check for agreement with the position errors predicted by Eq. 7.16.

Second, a computational method called the QR factorization can be used in the presence of redundant measurements to check for inconsistency among the measurements. An observable known as the parity vector describes the level of inconsistency. It has been shown that the errors predicted by Deines are different for each satellite. Thus, the double difference errors can be used to predict the magnitude of the parity vector in the presence of the predicted relativistic
effects, and can be compared with the parity vector from the ambiguity resolved solution at the end of the flight.

The ideal test would have been to fly straight and level at a constant velocity, with initial and final positions known to within 1-2 cm. This is not practical, however, so this profile was approximated as closely as possible. An accurate survey was made on the ramp at University Airport (UNI) in Albany, Ohio, followed by a direct 40 km flight to Rhodes Airport near Jackson, Ohio, where another survey was taken. This second survey served as both the end of the first flight test and the beginning of the second experiment which included the return flight. Thus, two data sets were collected. It is the return flight to UNI that is considered extensively here, because six satellites were available continuously during the flight. Six satellites were available during most of the first flight as well, but a few bad measurements were recorded for one satellite and therefore made the second data set more useful.

A description of the flight test is presented, followed by an extensive outline of how the predicted relativistic effects were calculated. The comparison between experimental and theoretical results is then made both in terms of position error and the parity vector.
7.4.2 Flight Test Description

On October 7, 1996, a flight test was conducted in Ohio University's Piper Saratoga (N8238C) Flying Laboratory. One passenger seat was removed and replaced by a palette which was bolted into place. A rack containing an Ashtech Z-12 GPS receiver was secured into place on the palette with a rachet tie-down. Two 12 volt batteries were connected to the rack, which can be set up internally as either a series (24 V) or parallel (12 V) connection. Before the flight, the batteries were connected in parallel to allow swapping of batteries without losing power to the receiver. A dual-frequency GPS antenna was secured on the top of the airplane and connected to the receiver via an antenna cable that fed directly into the cabin.

In the Avionics Hangar at University Airport (UNI), the Ground Station receiver was connected to an antenna in the crow's nest. The receiver remained in the GPS lab for the duration of the two flight tests. The Saratoga was pulled out to the front row of parking spaces on the ramp at UNI, and faced the runway. At approximately 15:27 GMT (11:27 AM local) both receivers began collecting data at an interval of once per second.

To obtain an accurate survey, a static data collection of about one hour was conducted. At approximately 16:31 GMT the engine was started and preparations were made for taxi and departure. The pilot took off at 16:39 GMT and climbed to about 1300 ft (~ 500 ft AGL) and initiated a left turn to a magnetic heading of 235 degrees. The destination was Rhodes Airport which is roughly 21 nmi (40 km) southwest of UNI. At 16:45 GMT the pilot executed a climb to 1700 ft to avoid a tower which was marked on the Ohio Aeronautical Chart at 1243 ft. At
16:51 GMT the pilot landed the airplane at Rhodes Airport. After taxiing to the parking area at Rhodes, the airplane was situated for another static data collection beginning at 16:54 GMT. Again data were collected for about one hour.

At 17:56 GMT the engine was started and takeoff for the return trip to UNI occurred at 18:01 GMT. The altitude for the return flight was approximately 2300 ft. Landing at UNI was delayed slightly due to inbound traffic. The pilot entered the pattern on the downwind leg and touchdown on Runway 25 took place at 18:16 GMT. At 18:20 GMT the airplane was parked and was again collecting static data. This final data collection concluded at 19:28 GMT.

7.4.3 Calculation of Expected Relativistic Effects

To numerically calculate the terms in Eq. 7.16, the ephemerides from the collected data were used to determine satellite positions during the flight test. Also, the beginning position at Rhodes Airport and the ending position at UNI were used to construct a straight and level flight profile between these two points. This was implemented as a constant rate of change in latitude and longitude with a constant ellipsoidal height of 500 m. The baseline was about 39.5 km and the duration of the simulated flight profile was 720 s yielding an approximate speed of 55 m/s for the aircraft, consistent with the ground speed of the Piper Saratoga during the flight test.

Using the aircraft position from the flight simulation and the satellite positions based on the received ephemerides, numerical first and second derivatives were taken to approximate the
velocity and acceleration vectors for the aircraft and also the velocity vector for the satellite. Thus, all the terms in Eq. 7.16 were known and the clock drifts predicted by Deines were calculated. See App. C for the MATLAB software used to calculate the predicted relativistic effects for this flight test.

The predicted relativity errors were calculated for each satellite-receiver pair. The results of the simulation for the ground and air receivers are shown in Tables 7.3 and 7.4, respectively. Here, the (SV#) term in the first column refers to one of the six satellites in view during the flight test. Recall that the relativistic effects predicted by Deines are different for each satellite.

According to Deines, the terms in Eq. 7.16 should be calculated using Earth-Centered Inertial (ECI) coordinates, then applied for users in the Earth-Centered Earth-Fixed (ECEF) frame. The ground receiver had a velocity and acceleration in the ECI frame due to Earth rotation, which is why the terms in Table 7.3 are nonzero. The third term is the same for all satellites because it depends only on receiver speed. The results show the accumulated effect after a simulated 12 minute flight from Rhodes Airport to University Airport.

It is not possible to observe errors of this magnitude without differencing techniques. To represent the error that would be observed in the carrier phase single difference (SD), the relativistic effects for each satellite were differenced between the ground and air receivers. The error would also be present in the code phase single differences, but the less noisy carrier phase measurements were used here. The single difference eliminates common satellite clock errors.
Table 7.3  Simulated Relativity Terms (Based on Deines) for the Ground Receiver after 12 Minutes Concurrent with the Flight from Rhodes Airport to University Airport

<table>
<thead>
<tr>
<th>GROUND</th>
<th>$\int_0^{720} \frac{A_G \cdot P_G}{c^2} dt$</th>
<th>$\int_0^{720} \frac{V_G \cdot v_i}{c^2} dt$</th>
<th>$\int_0^{720} \frac{-V_G^2}{2c^2} dt$</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>SV3</td>
<td>0.073 m</td>
<td>-0.913 m</td>
<td>-0.156 m</td>
<td>-0.996 m</td>
</tr>
<tr>
<td>SV9</td>
<td>-1.041 m</td>
<td>1.348 m</td>
<td>-0.156 m</td>
<td>0.151 m</td>
</tr>
<tr>
<td>SV17</td>
<td>-1.165 m</td>
<td>2.168 m</td>
<td>-0.156 m</td>
<td>0.847 m</td>
</tr>
<tr>
<td>SV23</td>
<td>-0.620 m</td>
<td>3.245 m</td>
<td>-0.156 m</td>
<td>2.469 m</td>
</tr>
<tr>
<td>SV26</td>
<td>-0.071 m</td>
<td>1.908 m</td>
<td>-0.156 m</td>
<td>1.682 m</td>
</tr>
<tr>
<td>SV28</td>
<td>-0.748 m</td>
<td>0.849 m</td>
<td>-0.156 m</td>
<td>-0.056 m</td>
</tr>
</tbody>
</table>

$i = 3, 9, 17, 23, 26, 28$

Table 7.4  Simulated Relativity Terms (Based on Deines) for Air Receiver after 12 Minutes of Straight and Level Flight from Rhodes Airport to University Airport

<table>
<thead>
<tr>
<th>AIR</th>
<th>$\int_0^{720} \frac{A_A \cdot P_A}{c^2} dt$</th>
<th>$\int_0^{720} \frac{V_A \cdot v_i}{c^2} dt$</th>
<th>$\int_0^{720} \frac{-V_A^2}{2c^2} dt$</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>SV3</td>
<td>0.256 m</td>
<td>-0.309 m</td>
<td>-0.197 m</td>
<td>-1.250 m</td>
</tr>
<tr>
<td>SV9</td>
<td>-1.417 m</td>
<td>1.801 m</td>
<td>-0.197 m</td>
<td>0.187 m</td>
</tr>
<tr>
<td>SV17</td>
<td>-1.474 m</td>
<td>2.186 m</td>
<td>-0.197 m</td>
<td>0.515 m</td>
</tr>
<tr>
<td>SV23</td>
<td>-0.749 m</td>
<td>3.603 m</td>
<td>-0.197 m</td>
<td>2.657 m</td>
</tr>
<tr>
<td>SV26</td>
<td>-0.281 m</td>
<td>2.250 m</td>
<td>-0.197 m</td>
<td>1.835 m</td>
</tr>
<tr>
<td>SV28</td>
<td>-0.773 m</td>
<td>0.641 m</td>
<td>-0.197 m</td>
<td>-0.328 m</td>
</tr>
</tbody>
</table>

$i = 3, 9, 17, 23, 26, 28$
Table 7.5 shows the results of differencing the predicted relativistic effects between the ground and air receivers. Again (SV#) refers to one of the six satellites in view during the flight test.

Because there are other error sources that are large in comparison to the predicted relativistic effects, further differencing was performed. Temperature effects, for instance, were shown in Chapter 6 to cause errors on the order of 10 meters in the single differences. Therefore, using atomic clocks to augment the receivers is not effective in trying to observe the relativistic effects. Instead, temperature effects and other common clock errors were removed by differencing against a reference satellite. Table 7.6 shows the predicted relativity errors for the carrier phase double differences. A high elevation satellite was chosen as the reference, in this case SV 17.

The double difference contains noise, multipath, and residual troposphere and ionosphere errors [Diggle, 1994]. For an L₁ ambiguity resolved carrier phase solution, these effects combine to produce position errors on the order of a few centimeters. Noise and multipath errors are small for carrier phase measurements, which are very clean compared to code phase measurements. Carrier phase advance through the ionosphere is approximately the same for receivers on a short baseline compared to the distance to the satellite. During the flight test, the maximum baseline was 40 km and even a satellite that passes directly overhead is still 20,200 km away. Thus, ionospheric errors are minor and diminish as the aircraft proceeds to University Airport where the ground station was located. Path delay through the troposphere depends on the altitude of the user, so we would expect an error to build up when the plane takes off and hold
Table 7.5  Simulated Relativity Terms (Based on Deines) after 12 Minute Flight from Rhodes Airport to University Airport for the Single Differences (Ground - Air)

<table>
<thead>
<tr>
<th>SD</th>
<th>Term 1</th>
<th>Term 2</th>
<th>Term 3</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>SV3</td>
<td>-0.183 m</td>
<td>0.396 m</td>
<td>-0.041 m</td>
<td>0.254 m</td>
</tr>
<tr>
<td>SV9</td>
<td>0.376 m</td>
<td>-0.454 m</td>
<td>-0.041 m</td>
<td>-0.037 m</td>
</tr>
<tr>
<td>SV17</td>
<td>0.309 m</td>
<td>-0.017 m</td>
<td>-0.041 m</td>
<td>0.332 m</td>
</tr>
<tr>
<td>SV23</td>
<td>0.129 m</td>
<td>-0.358 m</td>
<td>-0.041 m</td>
<td>-0.188 m</td>
</tr>
<tr>
<td>SV26</td>
<td>0.147 m</td>
<td>-0.342 m</td>
<td>-0.041 m</td>
<td>-0.154 m</td>
</tr>
<tr>
<td>SV28</td>
<td>0.025 m</td>
<td>0.207 m</td>
<td>-0.041 m</td>
<td>0.273 m</td>
</tr>
</tbody>
</table>

\[
\text{Term 1} = \int_{0}^{720} \frac{A_G \cdot \rho_{g_i} - A_A \cdot \rho_{A_i}}{c^2} dt
\]

\[
\text{Term 2} = \int_{0}^{720} \frac{V_G \cdot v_i - V_A \cdot v_j}{c^2} dt
\]

\[
\text{Term 3} = \int_{0}^{720} \frac{V_G^2 - V_A^2}{2c^2} dt
\]

\[i = 3, 9, 17, 23, 26, 28\]
Table 7.6  Simulated Relativity Terms (Based on Deines) After 12 Minute Flight From Rhodes Airport to University Airport for the Double Differences — SV 17 is the Reference

<table>
<thead>
<tr>
<th>DD</th>
<th>Term 1</th>
<th>Term 2</th>
<th>Term 3</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>SV17 - SV3</td>
<td>0.492 m</td>
<td>-0.413 m</td>
<td>0 m</td>
<td>0.079 m</td>
</tr>
<tr>
<td>SV17 - SV9</td>
<td>-0.067 m</td>
<td>0.436 m</td>
<td>0 m</td>
<td>0.369 m</td>
</tr>
<tr>
<td>SV17 - SV23</td>
<td>0.180 m</td>
<td>0.340 m</td>
<td>0 m</td>
<td>0.521 m</td>
</tr>
<tr>
<td>SV17 - SV26</td>
<td>0.162 m</td>
<td>0.324 m</td>
<td>0 m</td>
<td>0.486 m</td>
</tr>
<tr>
<td>SV17 - SV28</td>
<td>0.284 m</td>
<td>-0.225 m</td>
<td>0 m</td>
<td>0.060 m</td>
</tr>
</tbody>
</table>

Term 1 = \[ \int_{0}^{720} \left( \frac{A_{G} \cdot \rho_{G_{17}} - A_{A} \cdot \rho_{A_{17}}}{c^2} - \frac{A_{G} \cdot \rho_{G_{j}} - A_{A} \cdot \rho_{A_{j}}}{c^2} \right) dt \]

Term 2 = \[ \int_{0}^{720} \left( \frac{V_{G} \cdot v_{17} - V_{A} \cdot v_{17}}{c^2} - \frac{V_{G} \cdot v_{j} - V_{A} \cdot v_{j}}{c^2} \right) dt \]

Term 3 = \[ \int_{0}^{720} \left( \frac{V_{G}^2 - V_{A}^2}{2c^2} - \frac{\left(V_{G} - V_{A}\right)^2}{2c^2} \right) dt = 0 \]

\( j = 3, 9, 23, 26, 28 \)

140
approximately constant enroute. The error would drop out as the plane and the ground station
reach a state of similar altitude during landing. Note that tropospheric path delay is more
predictable than carrier phase advance through the ionosphere, and can therefore be modeled.
Thus, the error sources that impact the position solution cannot mask the relativistic effects
predicted in the double differences (Table 7.6), which are in some cases an order of magnitude
larger than the aforementioned error sources. Another potential error source is receiver inter-
channel biases, but these are generally calibrated by the manufacturer.

Table 7.6 shows that relativistic effects on the order of 0.5 meters can be expected for
certain double differences. These translate into position errors based on the geometry of the six
satellites under consideration. The time dilation term drops out of the double difference, as will
be shown in Sec. 7.4 and is not considered here — note that time dilation is a generally accepted
effect and has been verified experimentally [Hafele & Keating, 1972].

The first method of verifying the presence of relativistic effects is to determine the
predicted position error based on the accumulated effects after the 12 minute flight shown in
Table 7.6. The result is a horizontal position error of 0.473 meters and a vertical position error of
0.151 meters, which was determined as follows. From Diggle we have the form of the position
solution as calculated using double difference carrier phase measurements (assuming six
satellites are visible) [Diggle, 1994]:

\[ \text{position error} = \text{double difference} \times \text{factor} \]
\[
\begin{bmatrix}
DD^{12} - N^{12} & \\
DD^{13} - N^{13} & \\
DD^{14} - N^{14} & \\
DD^{15} - N^{15} & \\
DD^{16} - N^{16} & \\
\end{bmatrix} = 
\begin{bmatrix}
(u_x^1 - u_x^2) & (u_y^1 - u_y^2) & (u_z^1 - u_z^2) \\
(u_x^1 - u_x^3) & (u_y^1 - u_y^3) & (u_z^1 - u_z^3) \\
(u_x^1 - u_x^4) & (u_y^1 - u_y^4) & (u_z^1 - u_z^4) \\
(u_x^1 - u_x^5) & (u_y^1 - u_y^5) & (u_z^1 - u_z^5) \\
(u_x^1 - u_x^6) & (u_y^1 - u_y^6) & (u_z^1 - u_z^6) \\
\end{bmatrix} \begin{bmatrix}
x \\
y \\
z \\
\end{bmatrix}
\]

where: 
DD is the carrier phase double difference using satellites m and n
N is the combined integer ambiguity for the double difference
\( \lambda \) is the wavelength (19 cm for L_1 or 86 cm for L_1 - L_2)
\((u_x, u_y, u_z)\) is a unit vector pointing from the midpoint of the baseline between the two receivers to a satellite
\((x, y, z)\) is a vector representing the baseline from the ground receiver to the airborne receiver

This equation may be rewritten as:

\[
DD = H\beta
\]

where: 
DD contains carrier phase double difference measurements
H is a matrix of differenced unit vectors based on the satellite geometry
\( \beta \) is the \((x, y, z)^T\) baseline vector

To determine position error due to errors in the double differences, DD was replaced by a vector containing the predicted relativistic effects. Then a least squares solution for \( \beta \) yielded the position error due only to the relativistic effects. Here a snapshot of the geometry at the end of the 12 minute run time was used along with the built-up relativistic effects. First, the geometry matrix was formed using \((u_x, u_y, u_z)\) for each satellite as shown in Table 7.7. From these vectors
Table 7.7  Unit Vectors to Each Satellite from the Baseline Midpoint at the End of the Simulated Flight from Rhodes Airport to University Airport

<table>
<thead>
<tr>
<th>Geometry</th>
<th>$u_x$</th>
<th>$u_y$</th>
<th>$u_z$</th>
<th>Azimuth</th>
<th>Elevation</th>
</tr>
</thead>
<tbody>
<tr>
<td>SV3</td>
<td>-0.857</td>
<td>0.414</td>
<td>0.308</td>
<td>295.8°</td>
<td>17.9°</td>
</tr>
<tr>
<td>SV9</td>
<td>0.609</td>
<td>-0.286</td>
<td>0.740</td>
<td>115.1°</td>
<td>47.7°</td>
</tr>
<tr>
<td>SV17</td>
<td>0.067</td>
<td>-0.327</td>
<td>0.943</td>
<td>168.5°</td>
<td>70.5°</td>
</tr>
<tr>
<td>SV23</td>
<td>-0.152</td>
<td>0.365</td>
<td>0.918</td>
<td>337.4°</td>
<td>66.7°</td>
</tr>
<tr>
<td>SV26</td>
<td>0.670</td>
<td>0.564</td>
<td>0.483</td>
<td>49.9°</td>
<td>28.9°</td>
</tr>
<tr>
<td>SV28</td>
<td>-0.855</td>
<td>-0.331</td>
<td>0.400</td>
<td>248.8°</td>
<td>23.6°</td>
</tr>
</tbody>
</table>
we get azimuth and elevation angles which gives an indication of how the satellites are spaced in
the sky. SV 17 was chosen as the reference satellite in this case because it was highest in
elevation. The geometry matrix was formed using the differenced unit vectors shown in
Eq. 7.17.

Second, the relativistic errors from Table 7.6 were used to form the double difference
vector of Eq. 7.17, which can now be expressed:

\[
\begin{pmatrix}
0.079 \\
0.369 \\
0.521 \\
0.486 \\
0.060
\end{pmatrix}
\begin{pmatrix}
0.923 & -0.742 & 0.635 \\
-0.542 & -0.042 & 0.203 \\
0.219 & -0.693 & 0.024 \\
-0.603 & -0.892 & 0.459 \\
0.921 & 0.003 & 0.542
\end{pmatrix}
\begin{pmatrix}
err_x \\
err_y \\
err_z
\end{pmatrix}
\]

(7.19)

where the left hand side is a vector of relativistic effects, and the 5 x 3 H matrix represents the
satellite geometry. From this, a least squares solution was formed using the generalized inverse:

\[
\mathbf{\beta} = (H^T H)^{-1} H^T \mathbf{DD}
\]

(7.20)

This yielded (-0.198, -0.430, 0.151)^T for the x, y, and z position errors, or 0.473 m horizontal
error and 0.151 m vertical error. The three dimensional position error is about 0.5 m.

The second method of verifying the presence of relativistic effects is to form a parity
space residual. The residual exists when redundant measurements are available and is defined as
the norm of the parity vector. In this case, two redundant measurements were available which
resulted in a two element parity vector. A parity vector is formed by operating on H so that it is expressed as a product of an orthogonal matrix and an upper triangular matrix. Such a representation is called a QR decomposition, hence $H = QR$ and $Q^TQ = I$ [Golub and Van Loan, 1989]. This allows us to partition Eq. 7.18 in order to take advantage of the redundancy. This comes from the fact that the last $m-4$ rows of $R$ contain zeros where $m$ is the number of double difference measurements:

$$DD = H\hat{\beta}$$
$$DD = QR\hat{\beta}$$
$$Q^TDD = R\hat{\beta}$$

(7.21)

$$\begin{bmatrix}
Q_{p}^T \\
- - - \\
Q_{p}^T
\end{bmatrix}
\begin{bmatrix}
U \\
- - - \\
0
\end{bmatrix}
\begin{bmatrix}
\beta
\end{bmatrix}$$

The top partition yields the least squares solution as:

$$\hat{\beta} = U^{-1}Q_{\beta}^TDD$$

(7.22)

The bottom partition shows the following relation:

$$Q_{p}^TDD = 0$$

(7.23)

which is not true in general due to measurement noise, multipath, and other error sources. Thus, the result of Eq. 7.23 is known as the parity vector [Kline, 1991]:

145
In simple terms, the parity vector can be thought of as a measure of agreement among the different measurements. Thus, if inconsistencies exist in \( DD \) the parity residual will be large. It is important to note that if the double differences all have a common error, this will not show up in parity space. However, in the case of the predicted relativistic effects we have errors that are not common to each double difference. Therefore, disagreement among measurements should be observable in parity space.

The QR decomposition of \( H \) was performed:

\[
P = Q_p^T DD
\]  

(7.24)

\[
H = QR
\]

\[
\begin{bmatrix}
-0.595 & -0.475 & 0.120 & -0.182 & -0.610 \\
0.350 & -0.077 & 0.519 & -0.774 & 0.052 \\
-0.141 & -0.498 & -0.573 & -0.363 & 0.522 \\
0.389 & -0.717 & 0.323 & 0.470 & 0.102 \\
-0.594 & 0.080 & 0.533 & 0.121 & 0.585
\end{bmatrix}
\begin{bmatrix}
-1.551 & 0.176 & -0.454 \\
0 & 1.340 & -0.615 \\
0 & 0 & 0.605 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

(7.25)

The bottom partition of \( Q^T \) was used to form the parity vector:
The length of this vector, $|p| = 0.414$ m, shows that the residual should be more than 40 cm after the flight if the predicted relativistic effects were present in the data. During post processing, the predicted position error and parity space residual can be compared directly to what is observed when using the actual flight data.
7.4.4 Flight Test Results

Fig. 7.7 shows a plot of the ground track for the flight. The pilot took off on Runway 19 at Rhodes Airport and proceeded in a northeasterly heading to UNI. After entering the pattern on the downwind leg, the pilot landed on Runway 25 at UNI. Figs. 7.8-7.9 are plots of static data collections of 10 min duration taken before and after the flight. Fig. 7.8 shows the three dimensional position error on the ground at Rhodes airport, while Fig. 7.9 shows the three dimensional position error on the ground at UNI. This is determined by comparing the ambiguity resolved L\textsubscript{1} carrier phase solution from the FORTRAN software to the surveyed position as calculated by the PNAV software using the static data collections from Rhodes Airport and University Airport.

The position error at Rhodes Airport stayed mostly between 4 and 5 cm, or about a quarter wavelength on L\textsubscript{1}. The three dimensional position error may seem a bit large, but the baseline between the ground station and the Saratoga was 39.5 km during the static data collection at Rhodes Airport. Certain error sources, such as ionospheric carrier phase advance, have a tendency to decorrelate as the distance between air and ground platforms increases, resulting in larger position errors [Diggle, 1994]. It should be noted that ionospheric modeling was used by the PNAV software when the baseline was longer than 15 km, as was the case during the static collection at Rhodes Airport. No such correction was applied in the FORTRAN solution, yielding a larger position error for the 39.5 km baseline than for the 78 m baseline when the plane was parked at UNI after the flight. This is shown in Fig. 7.9 which indicates a position error of about 1-2 cm after the flight test. This represents a systematic error as ideally the
Figure 7.7  Ground Track for Flight from Rhodes Airport to University Airport
position error would have been zero-mean when the Saratoga was so close to the ground station. Certain error sources may not be zero-mean, such as multipath [van Nee, 1991]. However, this 1-2 cm offset was seen during the initial static collection at UNI as well, suggesting that multipath may not have been the cause. Still, Figs. 7.8-7.9 demonstrate a highly stable position solution that is accurate to within a few centimeters, clearly enough accuracy to measure effects on the order of 0.5 m.

The three-dimensional position error at the end of the flight was far less than the 0.5 m predicted by the MATLAB simulation based on the relativity terms taken from Eq. 7.16. Thus, the relativistic effects predicted by Deines are not supported by this flight test. Specifically, the first two terms of Eq. 7.16 have been shown to be in error. GPS receiver algorithms should not include corrections for these two terms.

To further verify the absence of the relativistic effects predicted by Deines, we turn to the second method proposed earlier in Sec. 7.4.3 — examination of the parity vector. The parity vector is plotted in Fig. 7.10 for two cases, one in which a troposphere model was used and one in which the troposphere model was turned off. It should be noted that Fig. 7.10 is not based on the PNAV solution from the Ashtech PRISM software package. The initial position at Rhodes Airport was fed to the FORTRAN software to fix the L₁ ambiguities. From there, the position solution and calculation of the parity vector are done in the FORTRAN software independent of the PNAV software. Recall that the norm of the parity vector, the parity space residual, is a measure of consistency among the measurements (in this case double differences). The parity

150
Figure 7.8  Static Collection Showing Initial 3D Position Error for 10 Minutes at Rhodes Airport Before the Flight to University Airport
Figure 7.9  Static Collection Showing Final 3D Position Error for 10 Minutes at University Airport After the Flight from Rhodes Airport
space residual based on the prediction of Deines is 0.414 m. In Fig. 7.10, the parity space residual does not exceed 0.08 m during the flight, and that is with the troposphere model turned off. This is clear evidence that the relativistic effects predicted by Deines were not present during the flight test.

The troposphere path delay is a particularly instructive phenomenon here because it represents an error source that depends on which satellite is being used — much like the $\frac{4 \cdot \mathbf{p}}{c^2}$ and $\frac{V \cdot \mathbf{v}}{c^2}$ terms proposed by Deines. When the troposphere model is turned off, the parity vector changes as the aircraft gains or loses altitude. When the plane goes higher the parity space residual goes up, and when the plane descends the parity space residual goes lower. The correlation is evident when Fig. 7.10 is compared with Fig. 7.11, a plot of altitude during the flight. Ultimately the plane lands at university airport and the effects of troposphere path delay are diminished.

An important characteristic of Fig. 7.10 is that the troposphere errors drop out when the plane lands at University Airport. This is in contrast to the predicted relativistic effects which were predicted to accumulate during the flight to a size which would cause the parity space residual to reach 0.4 m and not diminish. Thus, the predicted relativistic effects could not have caused the parity space residual to grow to 0.08 m and return to 0.01 m as in Fig. 7.10. To provide further proof, the parity space residual is also plotted (second curve in Fig. 7.10) for the case where the troposphere model is turned on. The model clearly removes most of the troposphere error as the parity space residual in this case stays at 0.03 m or less during the flight.
Figure 7.10  Parity Space Residual During Flight from Rhodes Airport to University Airport
Figure 7.11  Altitude (Ellipsoidal Height) During Flight from Rhodes Airport to University Airport
There is no indication of relativistic effects causing inconsistencies among the carrier phase measurements in Fig. 7.10. The parity space residual did not approach the 0.4 m level predicted. Thus, we can say once more that the relativistic effects predicted by Deines were not present in the flight data.

To show how sensitive the parity space residual is to disagreement among the measurements, a cycle slip was artificially injected in one of the carrier phase measurements (see Fig. 7.12). This was implemented by adding one cycle (19 cm) to the SV28 carrier phase measurement recorded by the airborne receiver, resulting in a jump of 11 cm in the parity space residual. This was done for each of the satellites, and jumps of anywhere from 9 to 16 cm were seen, showing the high degree of sensitivity of the parity space residual to errors on the order of magnitude predicted by Eq. 7.16.

For more proof, the predicted relativity effects were artificially injected as shown in Figs. 7.13-7.14. The relativity effects were implemented as linear drifts for 12 minutes, and they were subtracted (Fig. 7.13) from the carrier phase measurements. In Fig. 7.14 the relativistic effects were added to the carrier phase measurements. This was done in order to avoid doubling the effects in case they existed in the actual flight data. That is, if the relativistic effects were present the parity space residual would approximately double if the relativistic effects were added, and go closer to zero when the effects were subtracted. Because the parity space residual grew to about 40 cm in both cases, it is clear that the predicted relativistic effects did not occur in the original data.
Figure 7.12  Parity Space Residual During Flight from Rhodes Airport to University Airport with Artificially Injected Cycle Slip in SV 28
Figure 7.13  Parity Space Residual During Flight from Rhodes Airport to University Airport with Relativistic Effects Artificially Subtracted
Figure 7.14  Parity Space Residual During Flight from Rhodes Airport to University Airport with Relativistic Effects Artificially Added
7.5 Conclusions from Flight Test Experiment of Relativistic Effects

It has been shown that the relativistic errors predicted by Deines are not supported by experimental evidence. In particular, the \( \frac{A \cdot \rho}{c^2} \) and \( \frac{V \cdot v}{c^2} \) terms of Eq. 7.16 would cause large effects on the order of 0.5 m that were not observed in actual flight data. It is important that GPS receiver manufacturers do not include corrections for these two terms in position calculation algorithms.

The fact that these two terms were not measured during the flight leads us to wonder where the misinterpretation exists in the derivation of Deines. Nelson (1991) suggests that the Earth Centered Earth Fixed (ECEF), Earth Centered Inertial (ECI), and topocentric reference frames are equivalent as long as the appropriate corrections are made. It is possible that Deines made an error in applying a transformation between reference frames, though that question is left to the relativity experts.

Note, however, that the third term in the integral of Eq. 7.16, \( -\frac{V^2}{2c^2} \), drops out of the double difference and is not a part of the flight test results presented in this chapter. Note in Table 7.5 that time dilation caused a predicted error of -0.041 m during the flight that would be a component of the error in the single differences. However, stable clocks were not used during the flight test, and it would not be possible to verify the existence of the time dilation term based on the data collected. However, time dilation has been previously verified experimentally [Hafele & Keating, 1972] and would affect receivers that rely on clock-aided navigation. Note
that the GPS satellite clocks are corrected for time dilation as shown in the first integral of Eq. 7.16. There should be a similar correction for receivers on dynamic platforms. Thus, the time dilation term proposed by Deines for moving receivers is correct, but needs to be interpreted properly.

As an example, consider a low Earth orbit satellite (LEO) at an altitude of 300 km above the Earth's surface. By Kepler's Third Law the orbital period would be 5,431 seconds, or about 90.5 min. Assuming a circular orbit, the velocity of the LEO would be 7.726 km/s. The clock drift due to time dilation would be $-3.32 \times 10^{-10}$ s/s and the gravitational term, $\Delta \phi$, would be $0.31 \times 10^{-10}$ s/s. The combined clock drift is $-3.01 \times 10^{-10}$ s/s, or $-0.09$ m/s which is equivalent to almost half the L₁ wavelength per second. In only 100 s a 9 m error would build up. It is important to remember that double differencing would eliminate this error because it affects each single difference by the same amount. However, if clock coasting were used during periods of poor geometry, this error would have to be corrected.

It is intuitive that a GPS receiver at some altitude above the Earth would require a correction for gravitational potential as well. The effect is small for users on or near the Earth, but some applications might require a GPS receiver to be placed on a dynamic platform at a significant altitude. As an example, a clock at 39° N latitude with an altitude of 10,000 ft (3048 m) above the geoid (a reference surface on which ideal clocks beat at the same rate) experiences a clock drift of $-3.32 \times 10^{-13}$ (s/s). This is calculated by first approximating the effective gravity at that latitude [Ashby and Spilker, 1996]:

161
\[ g_{\text{eff}}(\cos \lambda) = 9.832099 - 0.051038(\cos \lambda)^2 - 0.000779(\cos \lambda)^4 \] (7.27)

where:
- \( g_{\text{eff}} \) is the effective gravity (m/s\(^2\))
- \( \lambda \) is the latitude

For \( \lambda = 39^\circ \), \( g_{\text{eff}} \) is 9.801 m/s\(^2\) which is used to determine the clock drift at 10,000 ft (3048 m):

\[ \frac{-g_{\text{eff}}}{c^2} h = \frac{-9.801 \text{(m/s}^2\text{)(3048 m)}}{c^2} \]

\[ = -3.32 \cdot 10^{-13} \text{(s/s)} \] (7.28)

This clock drift would cause an offset of -0.1 m after 1000 s (~17 min). Thus, even though this is a small term, it is potentially significant for clock-aided navigation. A correction for gravitational frequency shift is applied to the satellite clocks (Eq. 7.11), and it would be appropriate to evaluate this effect for a given airborne application to determine the size of the error and whether or not it can be ignored for a GPS receiver.

Intuitively, it seems that the relativistic effects for a moving GPS receiver should be satellite independent. The two terms in Eq. 7.16 that have been shown to be in error are different for each satellite. Ultimately, the determination of appropriate relativistic corrections is beyond the scope of this dissertation and is a question to be posed to relativity experts.